色々な不定積分

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問題

$a > 0$ をみたす実数とするとき以下の不定積分を計算せよ.

(1) $\displaystyle\int \frac{1}{x^2+a^2}\ dx$　(2) $\displaystyle\int \frac{1}{x^2-a^2}\ dx$　(3) $\displaystyle\int \frac{1}{a^2-x^2}\ dx$

(4) $\displaystyle\int \frac{1}{\sqrt{x^2+a^2}}\ dx$　(5) $\displaystyle\int \frac{1}{\sqrt{x^2-a^2}}\ dx$　(6) $\displaystyle\int \frac{1}{\sqrt{a^2-x^2}}\ dx$

(7) $\displaystyle\int \sqrt{x^2+a^2}\ dx$　(8) $\displaystyle\int \sqrt{x^2-a^2}\ dx$　(9) $\displaystyle\int \sqrt{a^2-x^2}\ dx$

解答

(1)

$x = a\tan t$ と置くと, $dx = a\sec^2 t$.

よって,

$$\begin{aligned}\int \frac{1}{x^2+a^2}\ dx &= \frac{1}{a^2}\int \cos^2 t \times a\sec^2 t\ dt \\ &= \frac{1}{a}\int\ dt \\ &= \frac{t}{a}\end{aligned}$$

ここで, $x = a\tan t \iff t = \arctan\dfrac{x}{a}$ より,

$$\therefore\quad \int \frac{1}{x^2+a^2}\ dx = \boxed{\frac{1}{a}\arctan\frac{x}{a}}$$

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(2)

$$\begin{aligned}
\int \frac{1}{x^2-a^2}\,dx &= \int \frac{1}{2a}\left(\frac{1}{x-a}-\frac{1}{x+a}\right)\,dx \\
&= \frac{1}{2a}\left(\log(x-a)-\log(x+a)\right) \\
&= \boxed{\frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|}
\end{aligned}$$

(3)

$$\begin{aligned}
\int \frac{1}{a^2-x^2}\,dx &= \int \frac{1}{2a}\left(\frac{1}{x+a}-\frac{1}{x-a}\right)\,dx \\
&= \frac{1}{2a}\left(\log(x+a)-\log(x-a)\right) \\
&= \boxed{\frac{1}{2a}\log\left|\frac{x+a}{x-a}\right|}
\end{aligned}$$

(4)
$x = a\sinh t$ と置くと, $dx = a\cosh^2 t$.
よって,

$$\begin{aligned}
\int \frac{1}{\sqrt{x^2+a^2}}\,dx &= \frac{1}{a}\int \mathrm{sech}\ {}^2 t \times a\cosh^2 t\ dt \\
&= \int\ dt \\
&= t
\end{aligned}$$

ここで, $x = a\sinh t \iff t = \mathrm{arcsinh}\ \frac{x}{a}$ より,

$$\therefore\quad \int \frac{1}{\sqrt{x^2+a^2}}\,dx = \boxed{\mathrm{arcsinh}\ \frac{x}{a}}$$

(5)
$x = a\cosh t$ と置くと, $dx = a\sinh^2 t$.
よって,

$$\begin{aligned}\int \frac{1}{\sqrt{x^2 - a^2}}\, dx &= \frac{1}{a}\int \operatorname{csch}^2 t \times a\sinh^2 t\ dt \\ &= \int\ dt \\ &= t\end{aligned}$$

ここで, $x = a\cosh t \iff t = \operatorname{arccosh}\dfrac{x}{a}$ より,

$$\therefore\quad \int \frac{1}{\sqrt{x^2 - a^2}}\, dx = \boxed{\operatorname{arccosh}\frac{x}{a}}$$

(6)
$x = a\sin t$ と置くと, $dx = a\cos t$.
よって,

$$\begin{aligned}\int \frac{1}{\sqrt{a^2 - x^2}}\, dx &= \frac{1}{a}\int \sec t \times a\cos t\ dt \\ &= \int\ dt \\ &= t\end{aligned}$$

ここで, $x = a\sin t \iff t = \arcsin\dfrac{x}{a}$ より,

$$\therefore\quad \int \frac{1}{\sqrt{a^2 - x^2}}\, dx = \boxed{\arcsin\frac{x}{a}}$$

(7)
$x = a\sinh t$ と置くと, $dx = a\cosh t$.
よって,

$$
\begin{aligned}
\int \sqrt{x^2+a^2}\ dx &= a\int \cosh t \times a\cosh t\ dt \\
&= a^2 \int \cosh^2 t\ dt \\
&= \frac{a^2}{2}\int (1+\cosh 2t)\ \ dt \\
&= \frac{a^2}{2}\left(t + \frac{1}{2}\sinh 2t\right) \\
&= \frac{a^2}{2}\left(t + \sinh t\ \ \cosh t\right) \\
&= \frac{a^2}{2}\left(t + \sinh t\ \sqrt{1+\sinh^2 t}\right)
\end{aligned}
$$

ここで, $x = a\sinh t \iff t = \mathrm{arcsinh}\ \dfrac{x}{a}$ より,

$$
\frac{a^2}{2}\left(t + \sinh t\ \sqrt{1+\sinh^2 t}\right) = \frac{a^2}{2}\left(\mathrm{arcsinh}\ \frac{x}{a} + \frac{x}{a}\ \sqrt{1+\left(\frac{x}{a}\right)^2}\right)
$$

$$
\begin{aligned}
\therefore \quad \int \sqrt{x^2+a^2}\ dx &= \frac{a^2}{2}\left(\mathrm{arcsinh}\ \frac{x}{a} + \frac{x}{a^2}\ \sqrt{a^2+x^2}\right) \\
&= \boxed{\frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\mathrm{arcsinh}\ \frac{x}{a}}
\end{aligned}
$$

(8)

$x = a\cosh t$ と置くと, $dx = a\sinh t$.

よって,

$$
\begin{aligned}
\int \sqrt{x^2 - a^2}\ dx &= a \int \sinh t \times a\sinh t\ dt \\
&= a^2 \int \sinh^2 t\ dt \\
&= \frac{a^2}{2} \int (-1 + \cosh 2t)\ \ dt \\
&= \frac{a^2}{2} \left(-t + \frac{1}{2}\sinh 2t \right) \\
&= \frac{a^2}{2} (-t + \sinh t\ \ \cosh t) \\
&= \frac{a^2}{2} \left(-t + \sqrt{\cosh^2 t - 1}\ \ \cosh t \right)
\end{aligned}
$$

ここで, $x = a\cosh t \iff t = \operatorname{arccosh} \dfrac{x}{a}$ より,

$$
\frac{a^2}{2} \left(-t + \sqrt{\cosh^2 t - 1}\ \ \cosh t \right) = \frac{a^2}{2} \left(-\operatorname{arccosh} \frac{x}{a} + \sqrt{\left(\frac{x}{a} - 1 \right)^2}\ \frac{x}{a} \right)
$$

$$
\begin{aligned}
\therefore \quad \int \sqrt{x^2 - a^2}\ dx &= \frac{a^2}{2} \left(-\operatorname{arccosh} \frac{x}{a} + \sqrt{\left(\frac{x}{a} - 1 \right)^2}\ \frac{x}{a} \right) \\
&= \boxed{\frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\operatorname{arccosh} \frac{x}{a}}
\end{aligned}
$$

(9)
$x = a\sin t$ と置くと, $dx = a\cos t$.
よって,

$$
\begin{aligned}
\int \sqrt{a^2 - x^2}\ dx &= a \int \cos t \times a\cos t\ dt \\
&= a^2 \int \cos^2 t\ dt \\
&= \frac{a^2}{2} \int (1 + \cos 2t)\ \ dt \\
&= \frac{a^2}{2} \left(t + \frac{1}{2}\sin 2t \right) \\
&= \frac{a^2}{2} \left(t + \sin t\ \ \cos t \right) \\
&= \frac{a^2}{2} \left(t + \sin t\ \sqrt{1 - \sin^2 t} \right)
\end{aligned}
$$

ここで, $x = a\sin t \iff t = \arcsin\dfrac{x}{a}$ より,

$$
\frac{a^2}{2} \left(t + \sin t\ \sqrt{1 - \sin^2 t} \right) = \frac{a^2}{2} \left(\arcsin\frac{x}{a} + \frac{x}{a}\ \sqrt{1 - \left(\frac{x}{a}\right)^2} \right)
$$

$$
\begin{aligned}
\therefore \quad \int \sqrt{a^2 - x^2}\ dx &= \frac{a^2}{2} \left(\arcsin\frac{x}{a} + \frac{x}{a^2}\ \sqrt{a^2 - x^2} \right) \\
&= \boxed{\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\arcsin\frac{x}{a}}
\end{aligned}
$$

研究

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \cosh x = \frac{e^x + e^{-x}}{2} \qquad \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\frac{d}{d\theta}\sin\theta = \cos\theta \qquad \frac{d}{d\theta}\cos\theta = -\sin\theta$$

$$\frac{d}{dx}\sinh x = \cosh x \qquad \frac{d}{dx}\cosh x = \sinh x$$

$$\sin^2\theta + \cos^2\theta = 1 \qquad \sinh^2 x + \cosh^2 x = -1$$

$$\mathrm{arcsinh}\ x = \log(x + \sqrt{x^2+1}) \qquad \mathrm{arccosh}\ x = \log(x + \sqrt{x^2-1})$$

$$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\arccos x = -\frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\arctan x = \frac{1}{1+x^2}$$

$$\frac{d}{dx}\mathrm{arcsinh}\ x = \frac{1}{\sqrt{1+x^2}} \qquad \frac{d}{dx}\mathrm{arccosh}\ x = \frac{1}{\sqrt{x^2-1}} \qquad \frac{d}{dx}\mathrm{arctanh}\ x = \frac{1}{1-x^2}$$

研究

$\int f(x)\ dx$ を計算する上で $x = g(t)$ とおくと, $dx = g'(t)\ dt$ より,

$$\int f(x)\ dx = \int f(g(t))g'(t)dt = \int \{F(g(t))\}'\ dt = F(g(t))$$

ここで $F(x)$ は $f(x)$ の原始関数を表す.
特に, $g(x)$ が $F(x)$ の逆関数となるとき,

$$\int f(x)\ dx = \int\ dt\ \left(= t = g^{-1}(x)\right)$$

となる.